Applications of Residue Calculus

Tuesday, March 16, 2021 11:00 AM

1) Computing integrals of the form

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$$

Substitute: $d\theta = -i\dfrac{de^{i\theta}}{e^{i\theta}}$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$-i\int_0^{2\pi} R\left(\frac{e^{i\theta}+e^{-i\theta}}{2}, \frac{e^{i\theta}-e^{-i\theta}}{2i}\right)\frac{de^{i\theta}}{e^{i\theta}} = \boxed{-i\oint_{C_1} R\left(\tfrac{1}{2}\left(z+\tfrac{1}{z}\right), \tfrac{1}{2i}\left(z-\tfrac{1}{z}\right)\right)\frac{dz}{z}}$$

$(C_1 = \{e^{i\theta}, 0 \le \theta \le 2\pi\})$.

compute it using residues if R is "nice" (rational!)

Example.

$$\int_{-\pi}^{\pi} \frac{\sin\theta}{1-2a\sin\theta+a^2}\, d\theta \qquad -1<a<1$$

$$-i\oint \frac{-\frac{i}{2}\left(z-\frac{1}{z}\right)}{1+ai\left(z-\frac{1}{z}\right)+a^2}\frac{dz}{z} =$$

$$-\frac{1}{2}\oint \frac{(z^2-1)\,dz}{z^2+aiz^3-aiz+a^2z^2} = -\frac{1}{2}\oint \frac{(z^2-1)\,dz}{z(aiz^2+(a^2+1)z-ai)} =$$

roots $z=0, ai, -\frac{i}{a}$

$$-\frac{1}{2}\cdot 2\pi i\left(\operatorname{Res}_{z=0} + \operatorname{Res}_{z=ai}\right) =$$

$$-\pi i\left(\frac{-1}{-ai} + \frac{(z^2-1)}{z(aiz^2+(a^2+1)z-ai)'}\Big|_{z=ai}\right) =$$

$$-\pi i\left(-\frac{i}{a} + \frac{-1-a^2}{ai(-2a^2+a^2+1)}\right) = \frac{\pi a}{1-a^2}.$$

$\operatorname{Res}_{z=0}\dfrac{f(z)}{z} = f(0)$

$f(z) = \dfrac{z^2-1}{aiz^2+(a^2+1)z-ai}$

If g has simple zero at z_0, $f(z_0)\neq 0$

$$\operatorname{Res}_{z=z_0}\frac{f}{g} = \frac{f(z_0)}{g'(z_0)}$$

$f(z) = \dfrac{z^2-1}{z}$

$g(z) = aiz^2+(a^2+1)z-ai$

$z_0 = ai$

2) $\displaystyle\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\,dx$, where P, Q - polynomials, $\deg Q \ge \deg P + 2$,

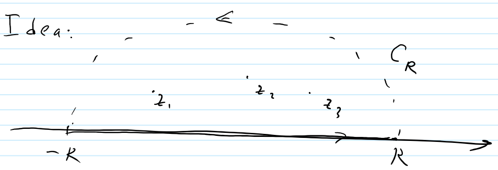
$Q(x)\neq 0$, $x\in\mathbb{R}$.

Improper integral converges: for large x,

$$\frac{P(x)}{Q(x)} = \frac{\sum_{k=1}^{m} a_k x^k}{\sum_{k=1}^{n} b_k x^k} = x^{m-n}\frac{\sum a_k x^{k-m}}{\sum b_k x^{k-n}} \sim x^{m-n} \le x^{-2}.$$

$n = \deg Q$, $m = \deg P$

Idea:



$C_R = \{Re^{it}, 0 \le t \le \pi\}$ - upper half-circle.

$\gamma_R = [-R, R] + C_R$. Choose R large: $R > \max\{|z| : Q(z)=0\}$.

Then

$$\oint_{\gamma_R}\frac{P(z)}{Q(z)}\,dz = 2\pi i\left(\sum_{\substack{Q(z_j)=0\\ \operatorname{Im} z_j>0}} \operatorname{Res}_{z=z_j}\frac{P(z)}{Q(z)}\right).$$

$$\| $$

$$\underbrace{\int_{-R}^{R}\frac{P(x)}{Q(x)}\,dx}_{\mathrm{I}} + \underbrace{\oint_{C_R}\frac{P(z)}{Q(z)}\,dz}_{\mathrm{II}}$$

As $R\to\infty$, $\mathrm{I} \to \displaystyle\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\,dx$

I $\qquad$ II

As $R\to\infty$, $\mathrm{I} \to \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$

$$|\mathrm{II}| \le \text{length}(C_R)\max_{|z|=R}\left|\frac{P(z)}{Q(z)}\right| \le 2\pi R\cdot C\frac{a_m/b_n}{R^2} \xrightarrow[R\to\infty]{} 0.$$

So, $$\boxed{\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{\substack{Q(z_j)=0\\ \operatorname{Im} z_j>0}} \operatorname{Res}_{z=z_j}\frac{P(z)}{Q(z)}}$$

Example

$$\int_{-\infty}^{\infty}\frac{x^2\,dx}{1+x^4} = 2\pi i\left(\sum_{\substack{z^4=-1\\ \operatorname{Im} z>0}}\operatorname{Res}_{z=z_j}\frac{z^2}{1+z^4}\right) = 2\pi i\left(\operatorname{Res}_{z=\frac{1+i}{\sqrt2}}\frac{z^2}{1+z^4} + \operatorname{Res}_{z=\frac{-1+i}{\sqrt2}}\frac{z^2}{1+z^4}\right) =$$

simple poles

$$2\pi i\left(\frac{z^2}{4z^3}\Big|_{z=\frac{1+i}{\sqrt2}} + \frac{z^2}{4z^3}\Big|_{z=\frac{i-1}{\sqrt2}}\right) = \frac{2\pi i}{4}\left(\frac{1-i}{\sqrt2}+\frac{-1-i}{\sqrt2}\right) = \frac{\pi}{2}\left(i\left(\frac{2(-i)}{\sqrt2}\right)\right) = \frac{\pi}{\sqrt2}$$

3) Integrals of the form

$$\int_{-\infty}^{\infty}\sin\lambda x\,\frac{P(x)}{Q(x)}\,dx \quad\text{and}\quad \int_{-\infty}^{\infty}\cos\lambda x\,\frac{P(x)}{Q(x)}\,dx$$

$\lambda>0$

where P, Q - polynomials,

$\deg Q \ge \deg P+1$, $Q(x)\ne 0$ for $x\in\mathbb{R}$.

Note: does not always converge absolutely:

if $\deg Q = \deg P+1$, $\frac{P(x)}{Q(x)} \sim \frac{1}{x}$ for large x, and $\int\frac{dx}{x}$ diverges!

$\cos\lambda x = \operatorname{Re} e^{i\lambda x}$

$\sin\lambda x = \operatorname{Im} e^{i\lambda x}$, so, if P and Q have real coefficients,

need to compute $\int_{-\infty}^{\infty} e^{i\lambda x}\frac{P(x)}{Q(x)}\,dx$.

Different approach than in Ahlfors: Jordan Lemma.

Camille Jordan

Theorem (Jordan Lemma).

Let f be a continuous function in the upper-half plane $\mathbb{H} := \{z : \operatorname{Im} z>0\}$. Let C_R be the semi-circle $\{|z|=R, \operatorname{Im} z>0\}$

$M_R := \sup_{z \in C_R} |f(z)|$. Assume $M(R) \to 0$ and $\lambda > 0$. Then

$$\int_{C_R} |f(z) e^{i\lambda z}| |dz| \xrightarrow[R\to\infty]{} 0 .$$

Direct "proof": $\cancel{\int_{C_R} |f(z) e^{i\lambda z}| |dz| \le M(R) \max_{C_R} |e^{i\lambda z}| \pi R = \pi R M(R) \overset{?}{\to} 0}$

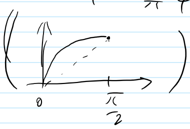
$\cancel{|e^{i\lambda(x+iy)}| = e^{-\lambda y} \le 1}$

Proof.
$$\int_{C_R} |f(z) e^{i\lambda z}| |dz| \le M_R \int_{C_R} |e^{i\lambda z}| |dz| = M_R \int_0^{\pi} |e^{i\lambda R(\cos\varphi + i\sin\varphi)}| R\, d\varphi =$$

$$M_R R \int_0^{\pi} e^{-\lambda R \sin\varphi} d\varphi = 2 M_R R \int_0^{\pi/2} e^{-\lambda R \sin\varphi} d\varphi \le$$

$\sin(\pi - \varphi) = \sin\varphi$

$\sin\varphi \ge \frac{2}{\pi}\varphi$ for $\varphi \in [0, \frac{\pi}{2}]$.



$$\le 2 M_R R \int_0^{\pi/2} e^{-\lambda R \frac{2}{\pi}\varphi} d\varphi = 2 M_R R \frac{\pi}{2R\lambda}\left(1 - e^{-\left(\frac{2\lambda R}{\pi}\cdot\frac{\pi}{2}\right)}\right) =$$

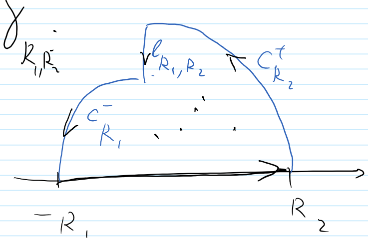
$$M_R \frac{\pi}{\lambda}(1 - e^{-\lambda R}) \le M_R \frac{\pi}{\lambda} \to 0 \ \blacksquare$$

Application of Jordan Lemma:

$$\int_{-\infty}^{\infty} e^{i\lambda x} \frac{P(x)}{Q(x)} dx = \lim_{R_1, R_2 \to \infty} \int_{-R_1}^{R_2} e^{i\lambda x} \frac{P(x)}{Q(x)} dx$$

$\cancel{\int_{-\infty}^{\infty} \frac{dx}{x} = 0 \qquad \int_{-\infty}^{\infty} x\, dx = \lim_{R\to\infty} \int_{-R}^{R} x\, dx = 0}$

γ_{R_1, R_2}:



Take R_1, R_2, such that all zeroes z of Q, $\operatorname{Im} z > 0$, satisfy $|z| < \min(R_1, R_2)$.

Then
$$\oint_{\gamma_{R_1,R_2}} e^{i\lambda z} \frac{P(z)}{Q(z)} dz = 2\pi i \sum_{\substack{Q(z_j)=0 \\ \operatorname{Im} z_j > 0}} \operatorname{Res}_{z = z_j} e^{i\lambda z} \frac{P(z)}{Q(z)}$$

But as $|z| \to \infty$ $\frac{|P(z)|}{|Q(z)|} \to 0$. So, by Jordan Lemma,

$$\left| \oint_{C^-_{R_1}} e^{i\lambda z} \frac{P(z)}{Q(z)} dz \right| \le \int_{C_{R_1}} \left| e^{i\lambda z} \frac{|P(z)|}{|Q(z)|} \right| |dz| \to 0$$

$$\left| \oint_{C^+_{R_2}} e^{i\lambda z} \frac{P(z)}{Q(z)} dz \right| \to 0$$

and $(R_1 \le R_2)$
$$\oint_{\ell_{R_1,R_2}} e^{i\lambda z} \frac{P(z)}{Q(z)} dz = \int_{R_1}^{R_2} e^{-\lambda y} \frac{P(iy)}{Q(iy)} dy \le \frac{e^{-\lambda R_1}}{\lambda} \underbrace{\max_{y \ge R_1} \frac{P(iy)}{Q(iy)}}_{\to 0} \to 0$$

So $\lim_{R_1, R_2 \to \infty} \int_{-R_1}^{R_2} e^{i\lambda x} \frac{P(x)}{Q(x)} dx$ exists. So

So $\lim\limits_{R_1,R_2\to\infty}\int_{-R_1}^{R_2} e^{i\lambda x}\frac{P(x)}{Q(x)}\,dx$ exists. So

$$\int_{-\infty}^{\infty}\cos\lambda x\,\frac{P(x)}{Q(x)}\,dx = \mathrm{Re}\Big(2\pi i\sum_{\substack{Q(z_j)=0\\ \operatorname{Im} z_j>0}}\operatorname{Res}_{z=z_j} e^{i\lambda z}\frac{P(z)}{Q(z)}\Big)$$

$$\int_{-\infty}^{\infty}\sin\lambda x\,\frac{P(x)}{Q(x)}\,dx = \mathrm{Im}\Big(2\pi i\sum_{\substack{Q(z_j)=0\\ \operatorname{Im} z_j>0}}\operatorname{Res}_{z=z_j} e^{i\lambda z}\frac{P(z)}{Q(z)}\Big)$$

both integrals converge!

Example.

$$\int_{-\infty}^{\infty}\frac{\sin\lambda x\cdot(x^3+1)}{x^4+1}\,dx = \mathrm{Im}\Big(2\pi i\Big(\operatorname{Res}_{z=\frac{1+i}{\sqrt2}}\Big(e^{i\lambda z}\frac{z^3+1}{z^4+1}\Big)+\operatorname{Res}_{z=\frac{-1+i}{\sqrt2}}\Big(e^{i\lambda z}\frac{z^3+1}{z^4+1}\Big)\Big)\Big)=$$

$$\mathrm{Im}\Big(2\pi i\Big(\Big(\frac{e^{i\lambda z}(z^3+1)}{4z^3}\Big)\Big|_{z=\frac{1+i}{\sqrt2}} + \Big(\frac{e^{i\lambda z}(z^3+1)}{4z^3}\Big)\Big|_{z=\frac{1+i}{\sqrt2}}\Big)\Big)=$$

$$\mathrm{Im}\Big(\frac{\pi i}{2}\Big(e^{i\lambda(\frac{1+i}{\sqrt2})}\Big(1-\frac{1+i}{\sqrt2}\Big)+e^{i\lambda(\frac{i-1}{\sqrt2})}\Big(1-\frac{1-i}{\sqrt2}\Big)\Big)\Big)=$$

$\frac{1}{z^3}=-z$ $\left(z=\frac{i+1}{\sqrt2}\right)$

$$\frac{\pi}{2}e^{-\lambda/\sqrt2}\Big(\cos\frac{\lambda}{\sqrt2}\Big(1-\frac{1}{\sqrt2}\Big)-\frac{\sin\frac{\lambda}{\sqrt2}}{\sqrt2}+\cos\frac{\lambda}{\sqrt2}\Big(1-\frac{1}{\sqrt2}\Big)-\frac{\sin\frac{\lambda}{\sqrt2}}{\sqrt2}\Big)=$$

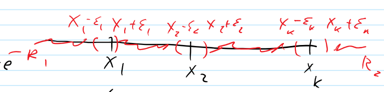
$$\boxed{\frac{\pi}{2}e^{-\lambda/\sqrt2}\cos\frac{\lambda}{\sqrt2}\,(2-\sqrt2)}$$

4) $\int_{-\infty}^{\infty}\cos\lambda x\,\frac{P(x)}{Q(x)}\,dx \quad \int_{-\infty}^{\infty}\sin\lambda x\,\frac{P(x)}{Q(x)}\,dx$, where $\deg Q>\deg P$, Q - has simple zeroes on $\mathbb{R}$.

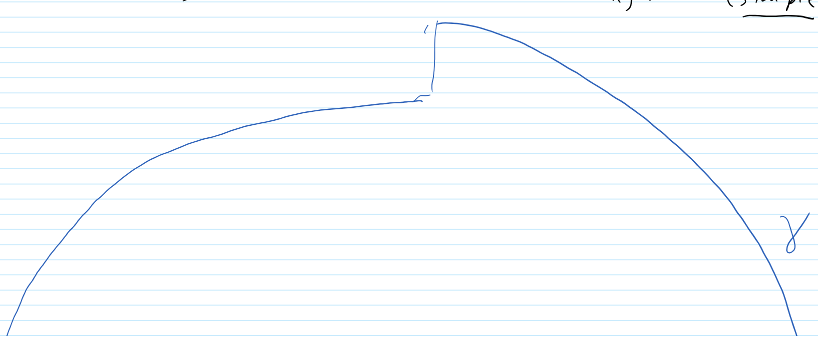
The integrals diverge near zeroes!
But we can consider principle value.

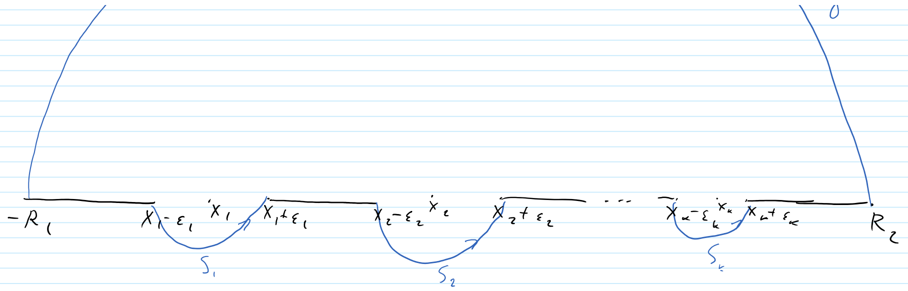
$$\text{p.v.}\int_{-\infty}^{\infty}\begin{matrix}\cos\lambda x\\ \sin\lambda x\end{matrix}\,\frac{P(x)}{Q(x)}\,dx = \lim_{\substack{R_1,R_2\to\infty\\ \varepsilon_1,\varepsilon_2,\dots\varepsilon_k\to0}}\int_{-R_1}^{x_1-\varepsilon_1}+\int_{x_1+\varepsilon_1}^{x_2-\varepsilon_2}+\dots\int_{x_k+\varepsilon_k}^{R_2}\dots$$

, where $x_1,\dots,x_k$ - zeroes of Q. (simple)



We remove symmetric intervals around x_j!





Observe: $\oint_\gamma e^{i\lambda z}\frac{P(z)}{Q(z)}dz = 2\pi i\Big(\sum_{\substack{Q(z_j)=0\\ \operatorname{Im} z_j>0}}\operatorname{Res}_{z=z_j}\frac{P(z)}{Q(z)}e^{i\lambda z} + \sum_{\substack{Q(x_j)=0\\ x_j\in\mathbb{R}}}\operatorname{Res}_{z=z_j}\frac{P(z)}{Q(z)}e^{i\lambda z}\Big).$

By Jordan Lemma, $\int_{C_R} \to 0.$

What about $\oint_{S_j} e^{i\lambda z}\frac{P(z)}{Q(z)}dz$?

x_j is a simple pole, so $e^{i\lambda z}\frac{P(z)}{Q(z)} = \frac{C_{-1}}{z-x_j}+g(z)$, where g is holomorphic at x_j. $C_{-1} = \operatorname{Res}_{z=z_j} e^{i\lambda z}\frac{P(z)}{Q(z)}$.

Then $\oint_{S_j} e^{i\lambda z}\frac{P(z)}{Q(z)}dz = \oint_{S_j}\frac{C_{-1}}{z-x_j}dz + \underbrace{\oint_{S_j} g(z)dz}_{\le \max|g(z)|\,\ell(S_j)\to 0,\ \varepsilon_j\to 0}$

$$I=\int_0^{\pi}\frac{C_{-1}}{(x_j-\varepsilon_j e^{it})-x_j}\,d(x_j-\varepsilon_j e^{it}) = \pi i\, C_{-1} = \pi i \operatorname{Res}_{z=x_j} e^{i\lambda z}\frac{P(z)}{Q(z)}.$$

So $\int_{S_j}\to \pi i \operatorname{Res}_{z=x_j} e^{i\lambda z}\frac{P(z)}{Q(z)}$

Plugging in the formula, we get

$$\text{p.v.}\int_{-\infty}^{\infty} e^{i\lambda x}\frac{P(x)}{Q(x)}dx = \pi i\Big(2\sum_{\substack{Q(z_j)=0\\ \operatorname{Im} z_j>0}}\operatorname{Res}_{z=z_j} e^{i\lambda z}\frac{P(z)}{Q(z)} + \sum_{\substack{Q(x_j)=0\\ x_j\in\mathbb{R}}}\operatorname{Res}_{z=z_j} e^{i\lambda z}\frac{P(z)}{Q(z)}\Big)$$

$$\text{p.v.}\int_{-\infty}^{\infty} \cos\lambda x\frac{P(x)}{Q(x)}dx = \operatorname{Re}\Big(\pi i\Big(2\sum_{\substack{Q(z_j)=0\\ \operatorname{Im} z_j>0}}\operatorname{Res}_{z=z_j} e^{i\lambda z}\frac{P(z)}{Q(z)} + \sum_{\substack{Q(x_j)=0\\ x_j\in\mathbb{R}}}\operatorname{Res}_{z=z_j} e^{i\lambda z}\frac{P(z)}{Q(z)}\Big)\Big)$$

$$\text{p.v.}\int_{-\infty}^{\infty} \sin\lambda x\frac{P(x)}{Q(x)}dx = \operatorname{Im}\Big(\pi i\Big(2\sum_{\substack{Q(z_j)=0\\ \operatorname{Im} z_j>0}}\operatorname{Res}_{z=z_j} e^{i\lambda z}\frac{P(z)}{Q(z)} + \sum_{\substack{Q(x_j)=0\\ x_j\in\mathbb{R}}}\operatorname{Res}_{z=z_j} e^{i\lambda z}\frac{P(z)}{Q(z)}\Big)\Big)$$

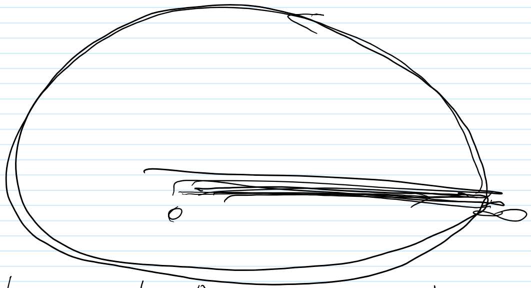
Example:

$$\text{p.v.}\int_{-\infty}^{\infty}\frac{\sin\lambda x}{1+x^3}dx = \operatorname{Im}\Big(\pi i\Big(2\operatorname{Res}_{z=e^{\frac{\pi i}{3}}}\frac{e^{i\lambda z}}{1+z^3} + \operatorname{Res}_{z=-1}\frac{e^{i\lambda z}}{1+z^3}\Big)\Big) =$$

$$\operatorname{Im}\Big(\pi i\Big(2\frac{e^{i\lambda(\frac12+\frac{\sqrt3}{2}i)}}{3(-\frac12+\frac{\sqrt3}{2}i)} + \frac{e^{-\lambda}}{3}\Big)\Big) =$$

$$\operatorname{Im}\Big(\pi i\Big(\frac23\Big(e^{-\frac{\sqrt3}{2}\lambda}\big(\cos\tfrac{\lambda}{2}+i\sin\tfrac{\lambda}{2}\big)\Big)\Big(-\tfrac12-\tfrac{\sqrt3}{2}i\Big)+\frac{e^{-\lambda}}{3}\Big)\Big) =$$

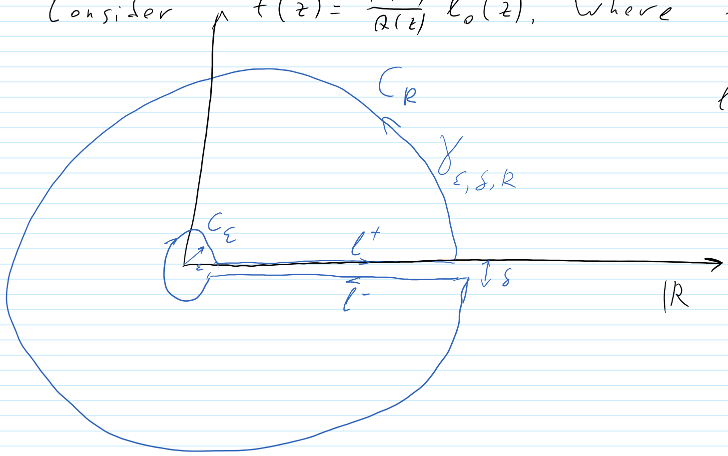
$$-\frac13\pi\Big(e^{-\frac{\sqrt3}{2}\lambda}\cos\frac{\lambda}{2} + \frac{\sqrt3}{3}e^{-\frac{\sqrt3}{2}\lambda}\sin\frac{\lambda}{2} + \frac{e^{-\lambda}}{3}\Big).$$



5) $\displaystyle\int_0^\infty \frac{P(x)}{Q(x)}\,dx$, $\deg Q \geq \deg P + 2$, $\forall x \geq 0$, $Q(x) \neq 0$.

Consider $f(z) = \dfrac{P(z)}{Q(z)}\, l_0(z)$, where $l_0(z) := \log|z| + \arg_0 z$, $0 \leq \arg_0 z < 2\pi$.

$l_0(z)$ - a branch of log discontinuous on $\mathbb{R}_+$.



Take ε, δ - small, R large, so that all $z: Q(z)=0$ lie inside the contour. Then

$$\oint_{\gamma_{\varepsilon,\delta,R}} f(z)\,dz = 2\pi i \Big(\sum_{Q(z_j)=0} \operatorname{Res}_{z=z_j} \frac{P(z)}{Q(z)}\, l_0(z) \Big)$$

But
$$\oint_{l^+} f(z)\,dz = \int_\varepsilon^R \frac{P(x)}{Q(x)}\, l_0(x)\,dx$$
$$\oint_{l^-} f(z)\,dz = -\int_\varepsilon^R \frac{P(x-i\delta)}{Q(x-i\delta)}\, l_0(x-i\delta)\,dx$$

$\boxed{\text{Fix } \varepsilon, R.}$ Let $\delta \to 0$. Then $\dfrac{P(x-i\delta)}{Q(x-i\delta)} \to \dfrac{P(x)}{Q(x)}$, uniformly on $[\varepsilon, R]$.

$l_0(x-i\delta) \to \underline{\underline{l_0(x) + 2\pi i}}$, also uniformly on $[\varepsilon, R]$.

So, as $\delta \to 0$, $\displaystyle\oint_{l^+} f(z)\,dz + \oint_{l^-} f(z)\,dz = \int_\varepsilon^R \frac{P(x)}{Q(x)}\, l_0(x)\,dx -$
$$2\pi i \int_\varepsilon^R \frac{P(x)}{Q(x)}\,dx - \int_\varepsilon^R \frac{P(x)}{Q(x)}\, l_0(x)\,dx =$$
$$-2\pi i \int_\varepsilon^R \frac{P(x)}{Q(x)}\,dx.$$

So $\displaystyle\oint_{C_R} f(z)\,dz - \oint_{C_\varepsilon} f(z)\,dz - 2\pi i \int_\varepsilon^R \frac{P(x)}{Q(x)}\,dx = 2\pi i \Big(\sum_{Q(z_j)=0} \operatorname{Res}_{z=z_j} \frac{P(z)}{Q(z)}\, l_0(z) \Big)$

So $\oint_{C_R} f(z)\,dz - \oint_{C_\varepsilon} f(z)\,dz - 2\pi i \int_\varepsilon^R \frac{P(x)}{Q(x)}\,dx = 2\pi i \left(\sum_{Q(z_j)=0} \operatorname{Res}_{z=z_j} \frac{P(z)}{Q(z)} \ell_0(z) \right)$

As $R \to \infty$

$$\left| \oint_{C_R} f(z)\,dz \right| \le 2\pi R \cdot \log R \underbrace{\max_{|z|=R} \frac{P(z)}{Q(z)}}_{\le \frac{C}{R^2}} \to 0.$$

As $\varepsilon \to 0$

$$\left| \int_{C_\varepsilon} f(z)\,dz \right| \le 2\pi\varepsilon \log \frac{1}{\varepsilon} \max_{|z|\le\varepsilon} \left| \frac{P(z)}{Q(z)} \right| \to 0.$$

So we get

$$\boxed{\int_0^\infty \frac{P(x)}{Q(x)}\,dx = -\left(\sum_{Q(z_j)=0} \operatorname{Res}_{z=z_j} \frac{P(z)}{Q(z)} \ell_0(z) \right)}$$

6) $\int_0^\infty x^\alpha \frac{P(x)}{Q(x)}\,dx \qquad 0<\alpha<1, \qquad \deg Q \ge \deg P + 2$ (to converge).
$Q(x) \ne 0 \quad x>0.$

As in 5): define $\ell_0(z)$, $z^\alpha := e^{\alpha \ell_0(z)}$.
Use the same contour for

$$f(z) = z^\alpha \frac{P(z)}{Q(z)}$$

$$\oint_{\gamma_{\varepsilon,\delta,R}} f(z)\,dz = 2\pi i \left(\sum_{Q(z_j)=0} \operatorname{Res}_{z=z_j} \frac{P(z)}{Q(z)} z^\alpha \right).$$

$$\oint_{\ell^+} f(z)\,dz = \int_\varepsilon^R x^\alpha \frac{P(x)}{Q(x)}\,dx$$

but $\lim_{\delta\to 0} \oint_{\ell^-} \underbrace{(x-i\delta)^\alpha}_{\downarrow \atop x^\alpha e^{2\pi i\alpha}} \frac{P(x-i\delta)}{Q(x-i\delta)}\,dx = -\int_\varepsilon^R e^{2\pi i\alpha} x^\alpha \frac{P(x)}{Q(x)}\,dx.$

So, since, as before, $\lim_{\varepsilon\to 0} \oint_{C_\varepsilon} f(z)\,dz = 0,$
$\lim$ [illegible]

So, since, as before, $\lim\limits_{\varepsilon \to 0} \oint\limits_{C_\varepsilon} f(z)\,dz = 0$,

$\lim\limits_{R \to \infty} \oint\limits_{C_R} f(z)\,dz = 0$

$$\int_0^\infty x^\alpha \frac{P(x)}{Q(x)}\,dx = \frac{2\pi i}{1-e^{2\pi i \alpha}} \left(\sum_{Q(z_j)=0} \mathrm{Res}_{z=z_j} \frac{P(z)}{Q(z)} z^\alpha \right).$$

$$\frac{\pi}{\sin \pi\alpha\, e^{\pi i \alpha}}$$

7) Let us consider

$\sum\limits_{n=0}^{\infty} \frac{P(n)}{Q(n)}$, where P, Q - polynomials, $\deg Q \geq \deg P + 2$, $Q(n) \neq 0$ ($n \in \mathbb{Z}$).

(Example: $\sum\limits_{n=0}^{\infty} \frac{1}{n^2+1}$)

Assume $\frac{P(z)}{Q(z)}$ - even $\left(\frac{P(-z)}{Q(-z)} = \frac{P(z)}{Q(z)} \right)$.

Or we can consider $\sum\limits_{n=-\infty}^{\infty} \frac{P(n)}{Q(n)}$.

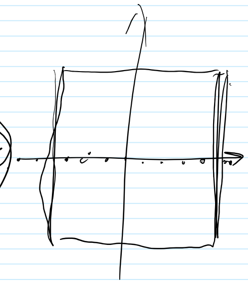
Consider $f(z) = \frac{P(z)}{Q(z)} \pi \cot\pi z$.

Observe: $\pi \cot \pi z = \pi i \left(\frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} \right) = \pi i \left(\frac{e^{2\pi i z} + 1}{e^{2\pi i z} - 1} \right)$

Singularities: at $z = n$, $n \in \mathbb{Z}$.

$\mathrm{Res}_{z=n} \frac{P(z)}{Q(z)} \pi \cot \pi z = \frac{\pi f(n) \cos \pi n}{(\sin \pi z)'|_{z=\pi n}} = \frac{P(n)}{Q(n)}$.

Other singularities of $f(z)$ - zeroes of Q.



Consider a square $S_N = \{ -N - \frac{1}{2} \leq \mathrm{Re}\, z \leq N + \frac{1}{2},\ -N - \frac{1}{2} \leq \mathrm{Im}\, z \leq N + \frac{1}{2} \}$

Observe that $w \to \frac{w+1}{w-1}$ maps 1 to ∞. It is one-to-one,

so the region $|w-1| > \delta$ is mapped to a bounded region (some disk). On ∂S_N: $|e^{2\pi i z} - 1| > 1 - e^{-\pi}$ (consider each side).

So $\exists M$: $|\pi \cot \pi z| \leq M \quad \forall z \in \partial S_N$.

So $\left| \oint\limits_{\partial S_N} f(z)\,dz \right| \leq \ell(\partial S_N) \cdot M \cdot \max\limits_{\partial S_N} \frac{P(z)}{Q(z)} \leq 4(N+1) M \cdot \frac{C}{N^2} \to 0$.

for some C.

So $\lim\limits_{N \to \infty} \frac{1}{2\pi i} \oint\limits_{\partial S_N} f(z)\,dz = \sum\limits_{z \in \mathbb{Z}} \overbrace{\mathrm{Res}_{z=n} f(z)}^{P(n)/Q(n)} + \sum\limits_{Q(z_j)=0} \mathrm{Res}_{z=z_j} f(z)$.

So $\sum\limits^{\infty} P(n)$ ...

So $\lim_{N\to\infty} \frac{1}{2\pi i}\oint_{\partial S_N} f(z)\,dz = \sum_{z\in\mathbb{Z}} \operatorname{Res}_{z=n} f(z) + \sum_{Q(z_j)=0} \operatorname{Res}_{z=z_j} f(z).$

So $\sum_{n=-\infty}^{\infty} \frac{P(n)}{Q(n)} = -\sum_{Q(z_j)=0} \operatorname{Res}_{z=z_j} f(z)$

For even $\frac{P(z)}{Q(z)}$:

$$\underbrace{\sum_{n=0}^{\infty} \frac{P(n)}{Q(n)}}_{} = \underbrace{\frac{1}{2}\sum_{n=-\infty}^{\infty} \frac{P(n)}{Q(n)}}_{} + \frac{1}{2}\frac{P(0)}{Q(0)} = \frac{1}{2}\frac{P(0)}{Q(0)} \cancel{+} \sum_{Q(z_j)=0} \operatorname{Res}_{z=z_j} f(z)$$

For our example, $z_1 = i$, $z_2 = -i$

$$\operatorname{Res}_{z=i} \frac{\pi \operatorname{cotan} \pi z}{z^2+1} = \frac{\pi}{2}\left(\frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}\right)$$

$$\operatorname{Res}_{z=-i} \frac{\pi \operatorname{cotan} \pi z}{z^2+1} = +\frac{\pi}{2}\left(\frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}\right)$$

So $\boxed{\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{\pi}{2}\left(\frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}\right) + \frac{1}{2}.}$